Optics of tunneling from adiabatic nanotapers

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This Letter develops a theory of light propagation along adiabatic photonic nanowire tapers having a diameter significantly less than the radiation wavelength $\lambda \sim 1$ μm, called nanotapers. The fundamental mode of a nanotaper primarily consists of an evanescent field, which propagates in the ambient medium and is very sensitive to the nanotaper shape. General analytical expressions for the evanescent field and the radiation loss of adiabatic nanotapers are obtained and applied to the investigation of the optics of tunneling from a nanotaper of a characteristic shape. As opposed to the effect of adiabatic bending, the radiation loss of a nanotaper occurs locally near a focal circumference of the evanescent field. In the latter case, the nanotaper is lossless everywhere except for that small location.

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If the optical microfiber diameter, d, is much less than the radiation wavelength, λ, then its fundamental mode primarily consists of the evanescent field that propagates in the ambient medium [1,2]. An optical microfiber with diameter $d \ll \lambda \sim 1$ μm is often called a photonic nanowire. The evanescent field around a nanowire and its radiation loss is very sensitive to nonuniformities. For this reason, practically, a nanowire can support the fundamental mode only if $d \gtrsim \lambda/10 \sim 100$ nm [3]. Transmission properties of an adiabatic nanowire taper with characteristic $d \sim 100$ nm (generally – with $d \ll \lambda$), called a nanotaper (NT) are the subject of this Letter.

The problem of radiation losses in tapered waveguides has been extensively studied [1,2]. However, to the author's knowledge, no analytical expressions for the radiation loss of tapers were obtained unless the variation of the taper's parameters were small and the perturbation theory was applicable (see [1,2] and discussion in [3]). In particular, a quite general problem of determination of the radiation loss of an adiabatic fiber taper remained unsolved. In the present Letter this problem is solved for the important case of a NT. A general expression for radiation loss is obtained. As an example, a NT with a characteristic Lorentzian variation of the propagation constant is investigated. The interesting optics of tunneling from this NT is demonstrated. It is shown all the radiation loss takes place in a small neighborhood of a focal circumference surrounding the NT.

For an axially symmetric NT, the scalar wave equation and the paraxial approximation can be applied. Then the field in the medium outside the NT, which has the refractive index n, can be expressed as $U(\rho,z)= \Psi(\rho,z)\exp(ikz)$, $k=2\pi n/\lambda$. Here the cylindrical coordinates (ρ,z,φ) are introduced and Ψ satisfies the Schrödinger equation $ik\Psi = -\Psi_{\rho\rho} - (1/\rho)\Psi_{\rho}$. Since the NT diameter $d(z)$ is a slow function of z, the following adiabatic solution is valid in the immediate vicinity of the NT:

$$U(\rho,z) \underset{|\gamma^{(0)}(z)|\rho\lesssim 1}{\approx} \left(\frac{2}{\pi k}\right)^{\frac{1}{2}} \gamma^{(0)}(z) K_0(|\gamma^{(0)}(z)|\rho) \exp\left[i\int^{z} dz\left(k - \frac{\gamma^{(0)}(z)^2}{2k}\right)\right] \tag{1}$$

where $K_0(x)$ is a modified Bessel function and $\gamma^{(0)}(z) = i|\gamma^{(0)}(z)|$ is a transversal component of the propagation constant k, which defines the characteristic inverse transversal size of the evanescent field. For a NT, $\gamma^{(0)}(z)$ is relatively small: $|\gamma^{(0)}(z)| << k$.

The adiabatic solution for evanescent field defined by Eq.(1) fails at $\rho \gtrsim L|\gamma^{(0)}(z)|/k$ where L $\sim |\gamma^{(0)}/(d\gamma^{(0)}/dz)|$ is a characteristic length of the NT nonuniformity. Eq.(1) can be continued to these ρ using the semiclassical approximation [4] as follows. Determine a parametric system of straight complex classical rays,

$$\rho(z,\gamma) = (\gamma/k)\left(z - z^{(0)}(\gamma)\right) \tag{2}$$

with parameter γ. Here $z^{(0)}(\gamma)$ is a function, which is inverse to $\gamma^{(0)}(z)$ and may be multi-valued. From Eq.(2), determine the function $\gamma(\rho,z)$ by expressing γ through ρ and z. Generally, the latter function is also multi-valued and has branches $\gamma_j(\rho,z)$. Let us denote the branch of function $z^{(0)}(\gamma)$ which corresponds to $\gamma_j(\rho,z)$ by the same index j: $z_j^{(0)}(\gamma)$. Then, the semiclassical solutions outside the NT can be constructed by sewing "the wave flesh" $U_j(\rho,z)$ on the "classical bones" $\gamma_j(\rho,z)$ as follows:

$$U_j(\rho,z) \underset{|\gamma_j^{(0)}(z)|\rho >> 1}{\approx} A_j(\rho,z)\exp(iS_j(\rho,z)) \tag{3}$$

where the amplitude $A_j(\rho,z)$ and the action $S_j(\rho,z)$ are expressed through the function $\gamma_j(\rho,z)$ as:

$$A_j(\rho,z)=\left(\frac{i\gamma}{k\rho}\right)^{1/2}\left[\frac{z-z_j^{(0)}(\gamma)}{\gamma(dz_j^{(0)}(\gamma)/d\gamma)}-1\right]^{-1/2}\Bigg|_{\gamma=\gamma_j(\rho,z)}, \quad (4)$$

$$S_j(\rho,z)=kz-\frac{\gamma^2}{2k}z+\gamma\rho+\frac{1}{k}\int^{\gamma}d\gamma\gamma z_j^{(0)}(\gamma)\Bigg|_{\gamma=\gamma_j(\rho,z)}. \quad (5)$$

From Eq.(2), for $\rho<<L|\gamma^{(0)}(z)|/k$, we have $\gamma_j(\rho,z)\approx\gamma_j^{(0)}(z)$ and the obtained solution coincides with Eq.(1) in the semiclassical region $|\gamma^{(0)}(z)|\rho>>1$.

By gluing together the solutions corresponding to different branches *j* in Eqs.(3)-(5) it is possible to determine a global solution for the evanescent field of a NT. Eqs. (3)-(5) are a particular case of WKB-Maslov solutions (generally, they can be expressed using the Maslov canonical operator) [4]. In [5] these solutions were applied to a quantum mechanical problem, which is mathematically similar to that considered in this Letter, – the evaluation of adiabatic transitions from the discrete to the continuous spectrum. The goal of [5] was to solve a scattering problem, while the intermediate characteristics of the wave function in the process of scattering were not studied. Here we are interested in finding the total radiation loss as well as understanding the optics of light propagation along a NT. For this reason, the method of contour integration in the plane of complex k^2 developed in [5,6], which solves the scattering problem by abstracting from the intermediate processes, is not sufficient for our purpose. Instead, we explore the Eqs. (2)-(5) directly.

Using Eqs.(2)-(5) it is straightforward to determine the total loss of a NT. Let us assume for simplicity (as in the example below) that there is only one term in the sum of Eq. (3) that contributes to the radiation loss and omit index *j*. From Eq.(2), for the radiation scattered at large z and ρ, we have $\gamma(\rho,z)\approx k\rho/z$. Then, integrating the flux of $U(\rho,z)$ at $z\to+\infty$ over ρ, we obtain for the radiation loss of a NT:

$$P=\frac{\pi^{\frac{3}{4}}}{4}\frac{k^{\frac{1}{2}}\left|\frac{dz^{(0)}}{d\gamma}\right|}{\left|\mathrm{Im}(z^{(0)}(0))\right|^{\frac{3}{2}}}\exp\left(-\frac{2}{k}\left|\mathrm{Im}\int_{\gamma_\infty}^{0}d\gamma\gamma z^{(0)}(\gamma)\right|\right) \tag{6}$$

This equation can also be found following the approach developed in [5]. It shows that $z^{(0)}(\gamma)$ is the only function necessary for determining the radiation loss in a NT. Eq. (6) dramatically simplifies and corrects the cumbersome expressions for radiation loss involving partial coupling coefficients [1,2]. It can be verified by integration by parts that the exponent in Eq.(6) coincides with the Landau-Dykhne formula for radiation loss [3]. The latter, however, misses the pre-exponent factor determined by Eq.(6).

Eq.(6) allows to find simple analytical expressions for NTs of different shapes. Consider, for example, a NT having the Lorentzian variation of the transversal propagation constant:

$$\gamma^{(0)}(z)=\gamma_\infty+\frac{\gamma_0-\gamma_\infty}{1+(z/L)^2} \tag{7}$$

Fig.1(a) compares this dependence with the corresponding NT diameter variation for characteristic parameters $k=4$ μm^{-1}, $\gamma_0=0.2i$ μm^{-1}, and $\gamma_\infty=0.4i$ μm^{-1} found numerically using the relations between d and $\gamma^{(0)}$ given in Ref. [1] . It is seen that only a small variation of $d(z)$ is needed to cause a significant variation of $\gamma^{(0)}(z)$. Calculation with Eqs.(6) and (7) yields:

$$\begin{aligned}
&P=\frac{\pi^{\frac{3}{2}}}{8}\left(\frac{k}{\gamma_\infty^2 L}\right)^{\frac{1}{2}}B(\delta)\exp\left(-\frac{L\gamma_\infty^2}{4k}f(\delta)\right),\\
&f(\delta)=(\delta+3)(\delta-1)\ln\left(\frac{1+\delta^{1/2}}{1-\delta^{1/2}}\right)+2\delta^{1/2}(3-\delta),\\
&B(\delta)=(1-\delta)\delta^{-5/4},\quad \delta=\gamma_0/\gamma_\infty.
\end{aligned} \tag{8}$$

To the author's knowledge, Eq. (8) is the first derived analytic expression for the radiation loss of a realistic fiber taper model. Fig. 1(b) shows the dependence of radiation loss, P, on the characteristic NT length, L, found from Eq.(8) for $\gamma_\infty = 0.4i\ \mu\text{m}^{-1}$ and different values of γ_0. It is seen that a very small variation of the NT diameter causes significant losses unless the characteristic length of the NT, L, is very large, of the order of millimeters. (compare Fig.1(a) and Fig.1(b) for $\gamma_0 = 0.2$). For a moderate NT diameter variation, when $d(\infty)-d(0)\sim d(\infty)$, we still have $|\gamma_0| << |\gamma_\infty|$ and the radiation loss is primarily determined by the NT shape near the center. In the latter case Eq.(8) simplifies:

$$P = (\pi^{3/2}/8)\Lambda^{-1/2}\exp(-8\Lambda/15), \qquad \Lambda = L\gamma_0^{5/2}/(k\gamma_\infty^{1/2}). \tag{9}$$

This result coincides, within a constant factor $\pi/2$, with the result for the parabolic model of $\gamma^{(0)}(z)$ found in quantum mechanics [6,7]. The difference in a constant factor is caused by different dimensionality of the problems considered here and in [6,7].

Let us now investigate the structure of the evanescent field and the optics of tunneling of light from a NT using, as an example, the NT defined by Eq.(7). Substitution of Eq.(7) into Eq.(2) yields the cubic equation for $\gamma(\rho,z)$:

$$\gamma^3(L^2+z^2)-\gamma^2(\gamma_0 L^2+\gamma_\infty z^2+2k\rho z)+\gamma(k^2\rho^2+2\gamma_\infty k\rho z)-\gamma_\infty k^2\rho^2=0 \tag{10}$$

which defines three branches of $\gamma(\rho,z)$. Only two of these branches continuously glued together (similar to the branches of a square root function in a complex plane) correspond to the boundary condition, Eq.(1), of our problem. As an illustration, Fig. 2 shows the corresponding two-sheet profile of dimensionless $\text{Re}(\gamma(\rho,z)/\gamma_\infty)$ in the plane of the dimensionless coordinates, $(\rho k/(L\gamma_\infty), z/L)$, calculated for γ_0/γ_∞=0.5. The branch point in Fig. 2, (ρ_b, z_b), is determined by the equations:

$$\rho_b = \frac{L|\gamma_\infty|}{4k}\sqrt{2Q - \sqrt{4Q^2 - \delta^3}}, \quad z_b = 0,$$
$$Q = \delta^2 + 18\delta - 27, \quad \delta = \gamma_0 / \gamma_\infty. \tag{11}$$

This branch point belongs to a complex caustic of the considered system of rays (recall that caustic is determined as a focal curve, i.e., a curve where the amplitude $A_j(\rho,z)$ in Eq.(3) turns to infinity). In the immediate neighborhood of caustic, the WKB solution, Eqs.(3) – (5), fails. Locally, near the branch point (ρ_b, z_b), we find that the caustic is determined by the equation $\rho \approx \rho_b + (\gamma(\rho_b,0)z/k)$, and the amplitude diverges as $A(\rho,z) \sim [\rho - \rho_b - (\gamma_b(\rho_b,0)z/k)]^{-1/4}$. An expression for $\gamma(\rho_b,0)$ derived from Eq. (11) (it is omitted here for briefness) shows that $\gamma(\rho_b,0)$ is imaginary, similar to γ_0 and γ_∞. Therefore, in a real plane (ρ,z) there is only one focal point, (ρ_b, z_b), where $A(\rho,z) = \infty$. This point is the intersection of the complex caustic curve with the real surface (ρ,z). In 3D space, (ρ,z,φ), the complex caustic curve corresponds to a complex caustic surface and the real point (ρ_b, z_b) corresponds to a real focal circumference.

The optics of light propagation along the NT can be further clarified using Fig. 2. The input NT mode is lunched from the left hand side of the upper sheet in Fig. 2. In the neighborhood of the focal point (ρ_b, z_b), the mode is split into the upper and lower parts. The lower part of the upper sheet corresponds to the field, which is guided by the NT, while the upper part contributes to the exponentially small radiation loss. At $z>0$, the lower (guiding) part of the mode is continued from the upper sheet to the region $\rho > \rho_b$ of the lower sheet. Similarly, at $z>0$, the upper (radiating) part of the mode is continued from the upper sheet to the region $\rho < \rho_b$ at the lower sheet. Thus, in the NT considered, the radiating part of the field splits off in the neighborhood of a focal point in the plane (ρ,z), corresponding to a focal circumference in a real space.

Generally, for adiabatic NTs, the focal circumferences, which coincide with intersections of complex caustic surfaces with real space, are responsible for radiation loss. In special cases, the complex caustics may be situated close to real space or become real. The latter situation is common for adiabatically bent waveguides where the caustics coincide with the surfaces of turning points of an effective potential barrier (see [1,8] and references therein). A similar situation will be demonstrated for a special model of a NT elsewhere [9]. Thus, a qualitative difference exists between the optics of tunneling from bent waveguides and from adiabatically tapered waveguides considered here. For the former, the radiation loss is usually distributed along the length of a waveguide. As opposed to that, the radiation loss of the latter is localized near focal circumferences and, therefore, adiabatic NTs are lossless away from these locations.

In summary, simple analytical expressions for the radiation loss of adiabatic NTs are obtained. The demonstrated existence of focal circumferences of the evanescent field clarifies the optics of mode propagation and tunneling of light from adiabatic NTs.

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Figure captions

Fig.1 (a) – Comparison of the NT transversal propagation constant variation defined by Eq. (7) and the corresponding NT diameter variation calculated using equations given in [1]. (b) – Radiation loss of a NT as a function of its characteristic length *L* calculated with Eq.(8). Parameters of Eqs.(7) and (8) are shown in the figure.

Fig. 2. The two-sheet surface plot of $\mathrm{Re}(\gamma_1(\rho,z)/\gamma_\infty)$.

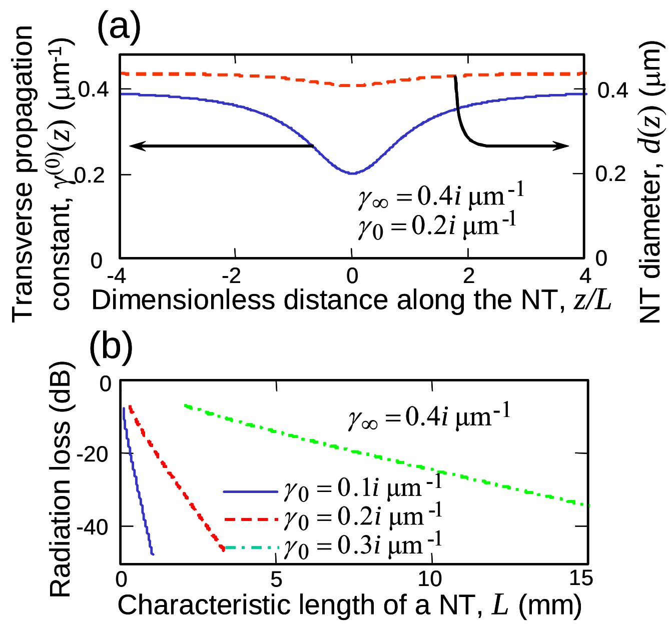
(a)
Transverse propagation constant, γ(0)(z) (μm-1)
NT diameter, d(z) (μm)
γ∞ = 0.4i μm-1
γ0 = 0.2i μm-1
Dimensionless distance along the NT, z/L
(b)
Radiation loss (dB)
γ∞ = 0.4i μm-1
γ0 = 0.1i μm-1
γ0 = 0.2i μm-1
γ0 = 0.3i μm-1
Characteristic length of a NT, L (mm)

Fig. 1

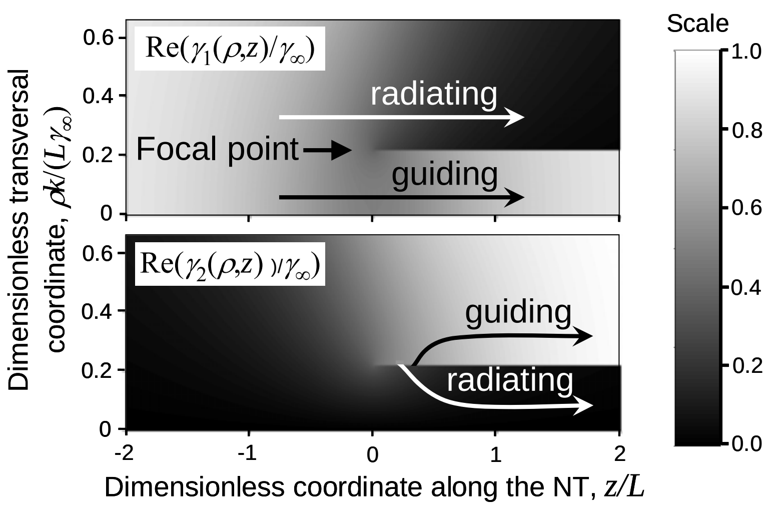
Re(γ1(ρ,z)/γ∞)
radiating
Focal point
guiding
Re(γ2(ρ,z))/γ∞)
guiding
radiating
Scale
1.0
0.8
0.6
0.4
0.2
0.0
0.6
0.4
0.2
0
-2
-1
0
1
2
Dimensionless transversal coordinate, ρk/(Lγ∞)
Dimensionless coordinate along the NT, z/L

Fig. 2